Math 155, ***Lecture Notes- Bonds*** **Name____________**

Section 9.1 Sequences

A **sequence** is a function whose domain is the set of positive integers. It will usually be denoted with subscript notation rather than function notation. You can use your graphing calculator in "sequence mode" to plot terms and create tables that show terms in a sequence.

For example:

$f(1) = a_1$ or $a(1) = a_1$

$f(2) = a_2$ or $a(2) = a_2$

$f(3) = a_3$ or $a(3) = a_3$

$\vdots$

$f(n) = a_n$ or $a(n) = a_n$

a_1 – first term

a_2 – second term

$\vdots$

a_n – n^{th} term

a_{n+1} – $(n+1)^{st}$ term

$\vdots$

An entire sequence can be denoted as $\{a_n\}$.

Ex. 1:

$$\{a_n\} = \left\{1 - \frac{1}{n}\right\} = \left\{0,\ \frac{1}{2},\ \frac{2}{3},\ \frac{3}{4},\ \frac{4}{5},\ \ldots\right\}$$

Ex. 2:

$$\{a_n\} = \{(-1)^n n\} = \{0,\ -1,\ 2,\ -3,\ 4,\ \ldots\}$$

Some sequences are *recursively defined*.

Ex. 3:

$\{d_n\}$ is defined as $d_{n+1} = d_n - 5$ and $d_1 = 25$.

So, $d_2 = d_1 - 5$

$d_2 = 25 - 5$

$d_2 = 20$

$d_3 = d_2 - 5$

$d_3 = 20 - 5$

$d_3 = 15$

$d_4 = d_3 - 5$

$d_4 = 15 - 5$

$d_4 = 10$

etc . . . - -

For the majority of the chapter, we'll be looking at sequences that have limiting values. These sequences are said to **converge**.

Ex. 4:

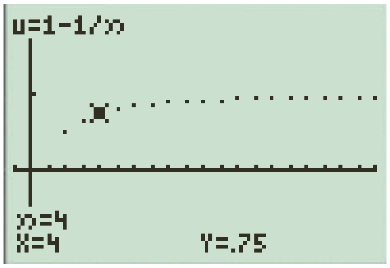
$$\{a_n\} = \left\{\frac{1}{2^n}\right\} = \left\{\frac{1}{2},\ \frac{1}{4},\ \frac{1}{8},\ \frac{1}{16},\ \frac{1}{32}, \ldots\right\}$$

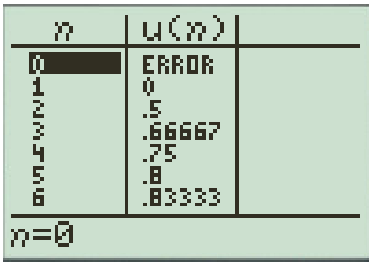
This sequence converges to 0.

Graphs, Tables and Ti- 83 Screen shots:

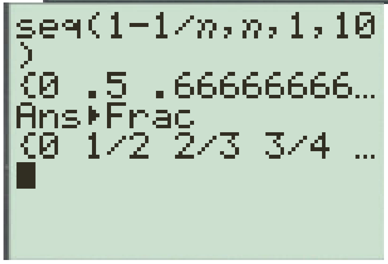
```
 Plot1  Plot2  Plot3
 nMin=1
\u(n)=1-1/n
 u(nMin)=
\v(n)=
 v(nMin)=
\w(n)=
 w(nMin)=
```

Ex. 1:



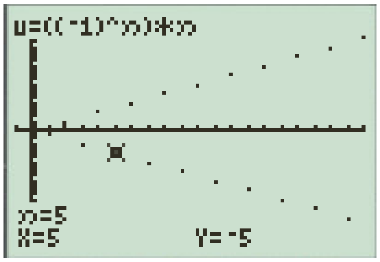


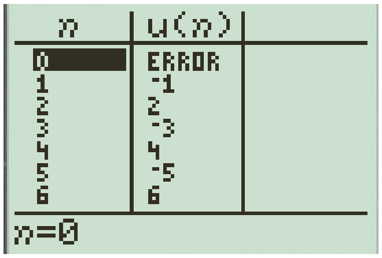
```
   n     u(n)
   0     ERROR
   1     0
   2     .5
   3     .66667
   4     .75
   5     .8
   6     .83333
n=0
```



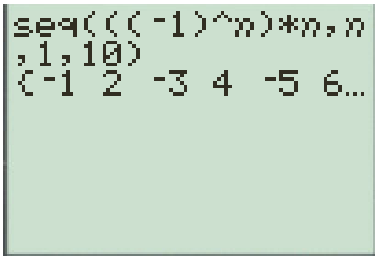
```
seq(1-1/n,n,1,10
)
{0 .5 .66666666…
Ans▸Frac
{0 1/2 2/3 3/4 …
```

Ex. 2:



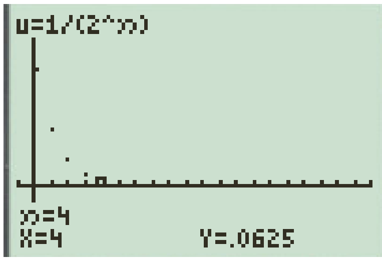


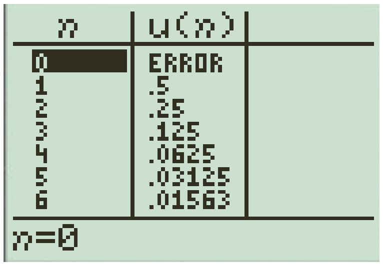
```
   n     u(n)
   0     ERROR
   1     ⁻1
   2     2
   3     ⁻3
   4     4
   5     ⁻5
   6     6
n=0
```



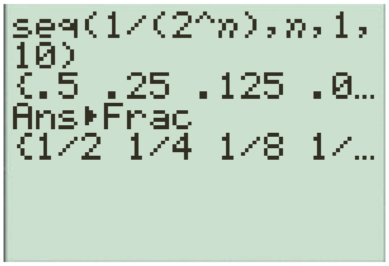
```
seq(((⁻1)^n)*n,n
,1,10)
{⁻1 2 ⁻3 4 ⁻5 6…
```

Ex. 4:





```
   n     u(n)
   0     ERROR
   1     .5
   2     .25
   3     .125
   4     .0625
   5     .03125
   6     .01563
n=0
```



```
seq(1/(2^n),n,1,
10)
{.5 .25 .125 .0…
Ans▸Frac
{1/2 1/4 1/8 1/…
```

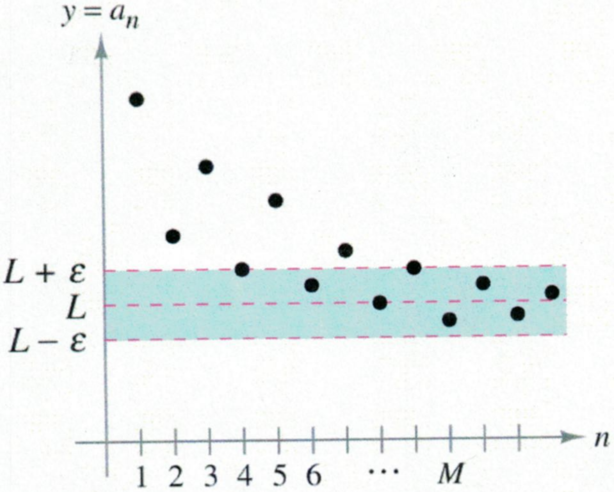
Definition of the Limit of a Sequence

Let L be a real number. The **limit** of a sequence $\{a_n\}$ is L, written as

$$\lim_{n\to\infty} a_n = L$$

if for each $\varepsilon > 0$, there exists $M > 0$ such that $|a_n - L| < \varepsilon$ whenever $n > M$. If the limit L of a sequence exists, then the sequence **converges** to L. If the limit of a sequence does not exist, then the sequence **diverges.**

If we plot the terms of a convergent sequence, we will see a "horizontal asymptote." That is, we will see the sequence exhibit asymptotic behavior.



Ex. 5:

Given: $\{a_n\} = \left\{\dfrac{n+4}{n+1}\right\}$

Consider $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \dfrac{n+4}{n+1}$

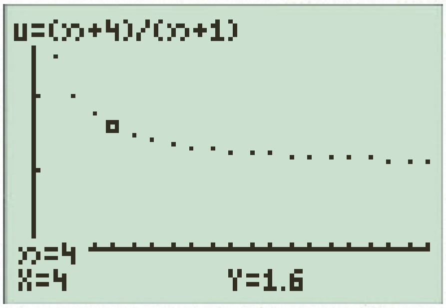
$$= \lim_{n\to\infty} \left[\frac{\frac{n+4}{1}}{\frac{n+1}{1}}\right] \cdot \left[\frac{\frac{1}{n}}{\frac{1}{n}}\right]$$

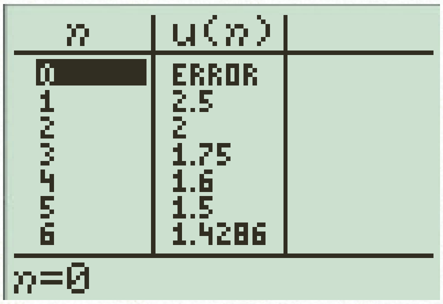
$$= \lim_{n\to\infty} \frac{1+\frac{4}{n}}{1+\frac{1}{n}}$$

$$= \frac{1}{1}$$

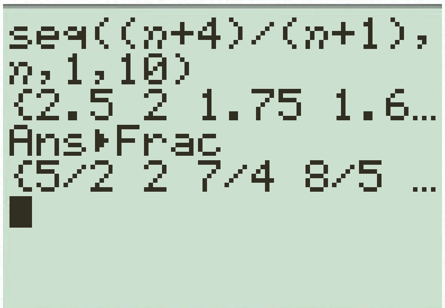
$$= 1 \checkmark$$

This sequence converges to 1.





```
 n  | u(n)
----+-------
 0  | ERROR
 1  | 2.5
 2  | 2
 3  | 1.75
 4  | 1.6
 5  | 1.5
 6  | 1.4286
n=0
```



```
seq((n+4)/(n+1),
n,1,10)
{2.5 2 1.75 1.6…
Ans▸Frac
{5/2 2 7/4 8/5 …
```

THEOREM 9.1 Limit of a Sequence

Let L be a real number. Let f be a function of a real variable such that

$$\lim_{x\to\infty} f(x) = L.$$

If $\{a_n\}$ is a sequence such that $f(n) = a_n$ for every positive integer n, then

$$\lim_{n\to\infty} a_n = L.$$

In other words, if a sequnce $\{a_n\}$ "agrees" with a function f at every positive integer, and if $f(x)\to L$ as $x\to\infty$, then $\{a_n\}\to L$ as well.

Ex. 6:

Given: $\{a_n\}=\left\{\left(1+\frac{1}{n}\right)^n\right\}$ Consider $\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x$

Let $y=\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x$

$$\ln(y)=\ln\left[\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x\right]$$

$$\ln(y)=\lim_{x\to\infty}\ln\left[\left(1+\frac{1}{x}\right)^x\right]$$

$$\ln(y)=\lim_{x\to\infty} x\cdot\ln\left(1+\frac{1}{x}\right)$$

$$\ln(y)=\lim_{x\to\infty}\left[\frac{x\cdot\left[\ln\left(1+\frac{1}{x}\right)\right]}{1}\right]\boxed{\frac{\frac{1}{x}}{\frac{1}{x}}}$$

$$\ln(y)=\lim_{x\to\infty}\frac{\ln\left(1+\frac{1}{x}\right)}{\frac{1}{x}} \longrightarrow \frac{\ln\left(1+\frac{1}{\infty}\right)}{\frac{1}{\infty}}=\frac{\ln(1)}{0}=\boxed{\frac{0}{0}}$$

Indeterminate form

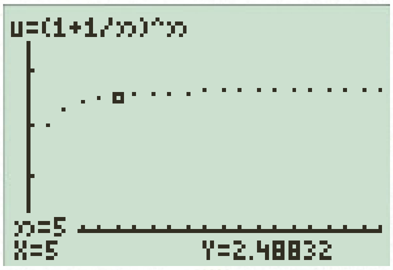
Use L'Hôpital's Rule

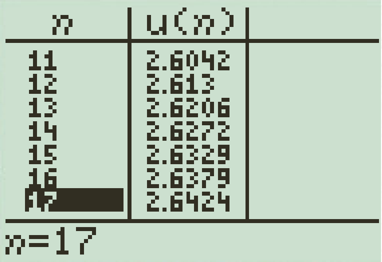
$$\ln(y)=\lim_{x\to\infty}\frac{\frac{d}{dx}\left[\ln\left(1+\frac{1}{x}\right)\right]}{\frac{d}{dx}\left[\frac{1}{x}\right]}$$

$$\ln(y)=\lim_{x\to\infty}\frac{\left(\frac{1}{1+\frac{1}{x}}\right)\cdot\left(-x^{-2}\right)}{\left(-x^{-2}\right)}$$

$$\ln(y)=\lim_{x\to\infty}\frac{1\ \left(-x^{-2}\right)}{1+\cancel{\frac{1}{x}}}$$

$$\ln(y)=1$$





n	u(n)
11	2.6042
12	2.613
13	2.6206
14	2.6272
15	2.6329
16	2.6379
17	2.6424

n=17

$$e^{\ln(y)}=e^1$$

$$y=e$$

So, $\lim_{n\to\infty} a_n = e$, and $\{a_n\}\to e$.

The sequence converges.

THEOREM 9.2 Properties of Limits of Sequences

Let $\lim_{n\to\infty} a_n = L$ and $\lim_{n\to\infty} b_n = K$.

1. $\lim_{n\to\infty} (a_n \pm b_n) = L \pm K$
2. $\lim_{n\to\infty} ca_n = cL$, c is any real number
3. $\lim_{n\to\infty} (a_nb_n) = LK$
4. $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{L}{K}$, $b_n \neq 0$ and $K \neq 0$

New Notation: Factorial !

Try working with these on your graphing calculator.

$n! = 1\cdot2\cdot3\cdot4 \cdots (n-1)\cdot n$

$0! = 1$

$3! = 1\cdot2\cdot3 = 6$

$5! = 1\cdot2\cdot3\cdot4\cdot5 = 120$

$2n! = 2(n!) = 2\cdot[1\cdot2\cdot3\cdot \cdots (n-1)\cdot n]$

$(2n)! = 1\cdot2\cdot3\cdot4 \cdots (n-1)\cdot n\,(n+1) \cdots (2n-1)\cdot 2n$

$4! = 1\cdot2\cdot3\cdot4 = 24$

$6! = 1\cdot2\cdot3\cdot4\cdot5\cdot6 = 720$

$2! = 1\cdot2 = 2$

$1! = 1 = 1$

THEOREM 9.3 Squeeze Theorem for Sequences

If

$$\lim_{n\to\infty} a_n = L = \lim_{n\to\infty} b_n$$

and there exists an integer N such that $a_n \le c_n \le b_n$ for all $n > N$, then

$$\lim_{n\to\infty} c_n = L.$$

Ex. 7:

Given: $\{a_n\} = \left\{\frac{\sin(n)}{n}\right\}$ Consider $\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(\frac{\sin(n)}{n}\right)$

We know $-1 \le \sin(n) \le 1$ for all $n > 0$.

So, we can see that $-\frac{1}{n} \le \frac{\sin(n)}{n} \le \frac{1}{n}$ for all $n > 0$.

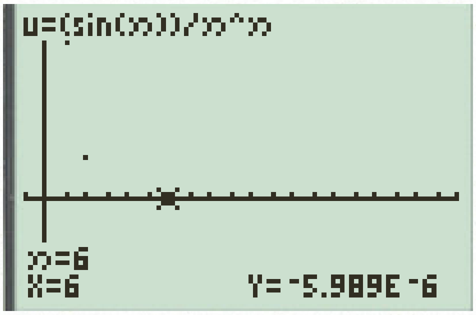
Since $\lim_{n\to\infty}\left(-\frac{1}{n}\right) = 0$ and $\lim_{n\to\infty}\left(\frac{1}{n}\right) = 0$, we can see that

$$\lim_{n\to\infty}\left(-\frac{1}{n}\right) \le \lim_{n\to\infty}\left(\frac{\sin(n)}{n}\right) \le \lim_{n\to\infty}\left(\frac{1}{n}\right)$$

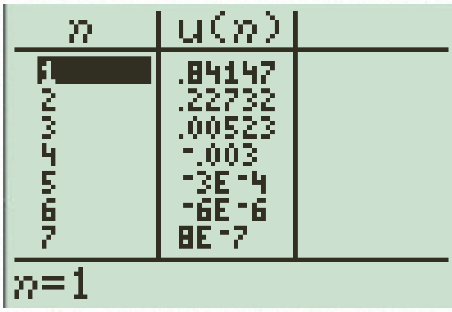
and $0 \le \lim_{n\to\infty}\left(\frac{\sin(n)}{n}\right) \le 0$.

This means that $\lim_{n\to\infty}\left(\frac{\sin(n)}{n}\right) = 0$, by the squeeze Theorem for Sequences, and $\{a_n\} \to 0$.

The sequence converges.



```
u=(sin(n))/n^n
n=6
X=6                Y=⁻5.989E⁻6
```

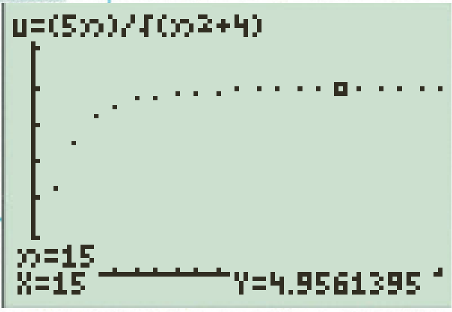


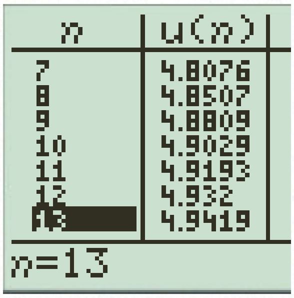
```
 n   | u(n)
-----+--------
 1   | .84147
 2   | .22732
 3   | .00523
 4   | ⁻.003
 5   | ⁻3E⁻4
 6   | ⁻6E⁻6
 7   | 8E⁻7
n=1
```

THEOREM 9.4 Absolute Value Theorem

For the sequence $\{a_n\}$, if

$$\lim_{n\to\infty} |a_n| = 0 \qquad \text{then} \qquad \lim_{n\to\infty} a_n = 0.$$





n	u(n)
7	4.8076
8	4.8507
9	4.8809
10	4.9029
11	4.9193
12	4.932
13	4.9419

n=13

Ex. 8:

Given: $\{a_n\} = \left\{\dfrac{5n}{\sqrt{n^2+4}}\right\}$ Consider $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \dfrac{5n}{\sqrt{n^2+4}}$

We know $n^2+4n+4 \geq n^2+4 \geq n^2$, for all $n \geq 1$.

we can see $(n+2)^2 \geq n^2+4 \geq n^2$, and

$\sqrt{(n+2)^2} \geq \sqrt{n^2+4} \geq \sqrt{n^2}$, by using algebra.

Now, we have $n+2 \geq \sqrt{n^2+4} \geq n$. Understanding properties of fractions, we can see $\dfrac{1}{n+2} \leq \dfrac{1}{\sqrt{n^2+4}} \leq \dfrac{1}{n}$. Using multiplication, we can see $\boxed{\dfrac{5n}{n+2} \leq \dfrac{5n}{\sqrt{n^2+4}} \leq \dfrac{5n}{n}}$. Applying Long Division, we have

$$\begin{array}{r} 5 \\ n+2\,\overline{)\,5n+0} \\ -5n-10 \\ \hline -10 \end{array} \quad \text{and} \quad \frac{5n}{n+2} = 5 - \frac{10}{n+2}.$$

This gives us $\boxed{5 - \dfrac{10}{n+2} \leq \dfrac{5n}{\sqrt{n^2+4}} \leq 5}$ ★★. As we apply the Squeeze Theorem, we can see

$$\lim_{n\to\infty}\left[5 - \frac{10}{n+2}\right] \leq \lim_{n\to\infty}\left[\frac{5n}{\sqrt{n^2+4}}\right] \leq \lim_{n\to\infty} 5, \text{ and}$$

$$5 \leq \lim_{n\to\infty} \frac{5n}{\sqrt{n^2+4}} \leq 5.$$ So, we have

$$\lim_{n\to\infty} \frac{5n}{\sqrt{n^2+4}} = 5$$, and the sequence converges.

Ex. 9:

Given: $\{a_n\} = \left\{ \frac{(n-2)!}{n!} \right\}$ Consider $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \frac{(n-2)!}{n!}$

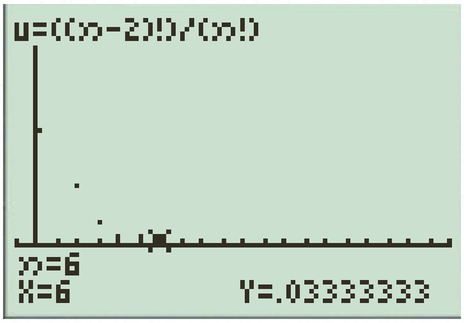
$$= \lim_{n\to\infty} \frac{1\cdot 2\cdot 3\cdot 4 \cdots \cancel{(n-4)}\cancel{(n-3)}\cancel{(n-2)}}{1\cdot 2\cdot 3\cdot 4 \cdots \cancel{(n-4)}\cdot\cancel{(n-3)}\cancel{(n-2)}(n-1)\cdot n}$$

$$= \lim_{n\to\infty} \frac{1}{(n-1)\cdot n}$$

$$= \lim_{n\to\infty} \frac{1}{n^2-n}$$

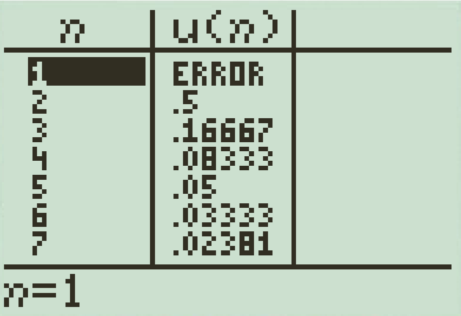
$$= 0$$

So, $\{a_n\} \to 0$, and the sequence converges.



u=((n-2)!)/(n!)

n=6
X=6 Y=.03333333



n	u(n)
1	ERROR
2	.5
3	.16667
4	.08333
5	.05
6	.03333
7	.02381

n=1

Ex. 10:

Given: $\{a_n\}=\left\{\frac{n^2}{2n+1}-\frac{n^2}{2n-1}\right\}$ Consider $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty}\left[\frac{n^2}{2n+1}-\frac{n^2}{2n-1}\right]$

$$=\lim_{n\to\infty}\left[\frac{n^2\cdot(2n-1)}{(2n+1)(2n-1)}-\frac{n^2\cdot(2n+1)}{(2n-1)(2n+1)}\right]$$

$$=\lim_{n\to\infty}\left[\frac{2n^3-n^2-2n^3-n^2}{(2n+1)(2n-1)}\right]$$

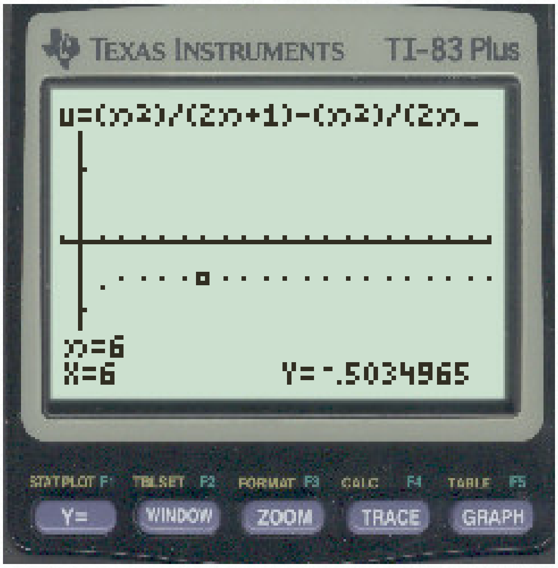
$$=\lim_{n\to\infty}\left(\frac{-2n^2}{4n^2-1}\right)$$

$$=\lim_{n\to\infty}\left[\frac{\frac{-2n^2}{1}}{\frac{4n^2-1}{1}}\right]\cdot\left[\frac{\frac{1}{n^2}}{\frac{1}{n^2}}\right]$$

$$=\lim_{n\to\infty}\frac{-2}{4-\frac{1}{n^2}}$$

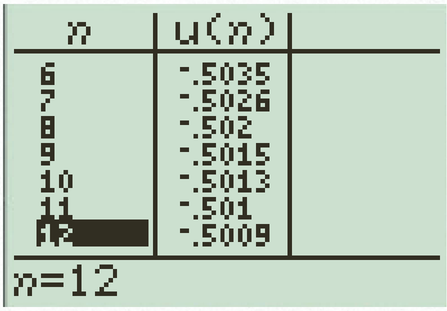
$$=\frac{-2}{4-0}$$

$$=\frac{-1}{2}$$



```
u=(n2)/(2n+1)-(n2)/(2n_
n=6
X=6              Y=-.5034965
```

So, $\{a_n\}\to\frac{-1}{2}$, and the sequence converges.



n	u(n)
6	-.5035
7	-.5026
8	-.502
9	-.5015
10	-.5013
11	-.501
12	-.5009

n=12

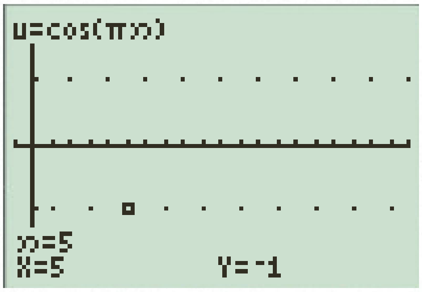
Ex. 11:

Given: $\{a_n\}=\{\cos(\pi n)\}$ Consider $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \cos(\pi n)$

We have $\{\cos(\pi n)\} = \{-1, 1, -1, 1, -1, 1, -1, 1, \ldots\}$

Since $\cos(\pi n)$ oscillates between -1 and 1, the $\lim_{n\to\infty} \cos(\pi n)$ does not exist.

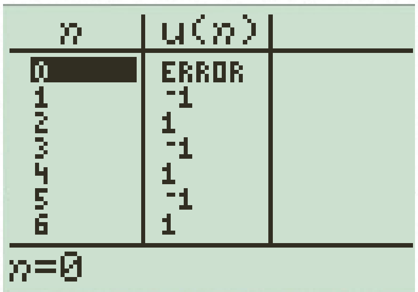
This means that $\{\cos(\pi n)\}$ diverges.



u=cos(πn)

n=5

X=5 Y=-1



n	u(n)
0	ERROR
1	-1
2	1
3	-1
4	1
5	-1
6	1

n=0

Ex. 12:

Given: $\{a_n\} = \left\{\frac{1\cdot 3\cdot 5\cdot 7\cdots(2n-1)}{n!}\right\}$ Consider $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \frac{1\cdot3\cdot5\cdot7\cdots(2n-1)}{n!}$

we need to better understand the ratio

$$\frac{1\cdot3\cdot5\cdot7\cdots(2n-1)}{n!} = \frac{1\cdot3\cdot5\cdot7\cdots(2n-1)}{1\cdot2\cdot3\cdot4\cdots(n)}$$

Consider $$\frac{1\cdot3\cdot5\cdot7\cdots(2n-1)}{1\cdot\underbrace{2\cdot3\cdot4\cdots(n)}_{n-1\text{ factors}}}$$

Is

$$\frac{3}{2}, \frac{5}{3}\geq\frac{3}{2}, \frac{7}{4}\geq\frac{3}{2}, \quad \frac{2n-1}{n}\geq\frac{3}{2}\,??$$

Solve for n: $\frac{2n-1}{n}\geq\frac{3}{2}$

$$2\cdot(2n-1)\geq 3\cdot n$$
$$4n-2\geq 3n$$
$$n\geq 2$$

So, for all $n\geq 2$, $\frac{2n-1}{n}\geq\frac{3}{2}$.

Therefore, $\frac{1\cdot3\cdot5\cdot7\cdots(2n-1)}{1\cdot2\cdot3\cdot4\cdots(n)}\geq\left(\frac{3}{2}\right)^{n-1}$.

Since $\lim\limits_{n\to\infty}\left(\frac{3}{2}\right)^{n-1}=\infty$, we have use the Squeeze Theorem to see

$$\lim_{n\to\infty}\frac{1\cdot3\cdot5\cdot7\cdots(2n-1)}{n!}\geq\lim_{n\to\infty}\left(\frac{3}{2}\right)^{n-1}=\infty$$

and $$\lim_{n\to\infty}\frac{1\cdot3\cdot5\cdot7\cdots(2n-1)}{n!}=\infty.$$

So, $\left\{\frac{1\cdot3\cdot5\cdot7\cdots(2n-1)}{n!}\right\}$ diverges.

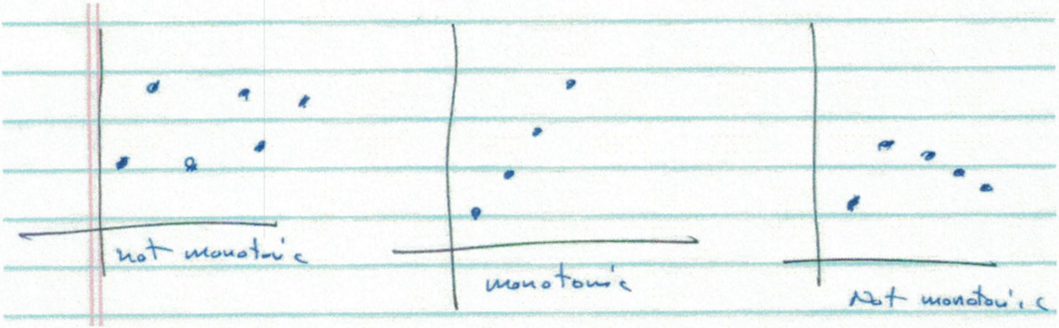
Definition of a Monotonic Sequence

A sequence $\{a_n\}$ is **monotonic** if its terms are nondecreasing

$$a_1 \leq a_2 \leq a_3 \leq \cdots \leq a_n \leq \cdots$$

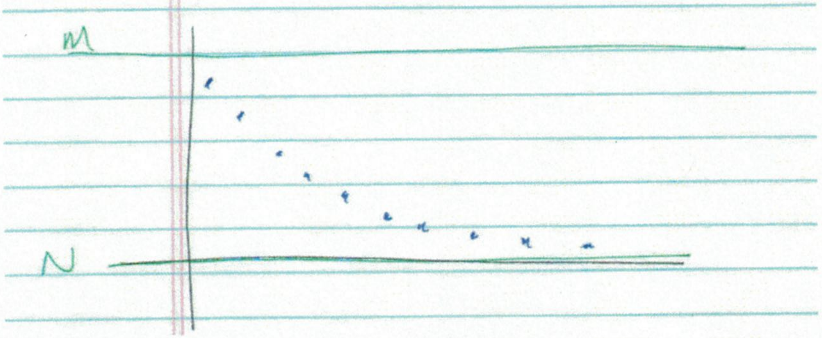
or if its terms are nonincreasing

$$a_1 \geq a_2 \geq a_3 \geq \cdots \geq a_n \geq \cdots.$$



Definition of a Bounded Sequence

1. A sequence $\{a_n\}$ is **bounded above** if there is a real number M such that $a_n \leq M$ for all n. The number M is called an **upper bound** of the sequence.
2. A sequence $\{a_n\}$ is **bounded below** if there is a real number N such that $N \leq a_n$ for all n. The number N is called a **lower bound** of the sequence.
3. A sequence $\{a_n\}$ is **bounded** if it is bounded above and bounded below.



THEOREM 9.5 Bounded Monotonic Sequences

If a sequence $\{a_n\}$ is bounded and monotonic, then it converges.

Ex. 13:

Given: $\{a_n\} = \left\{ne^{\frac{-n}{2}}\right\}$ Consider $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} ne^{\frac{-n}{2}}$

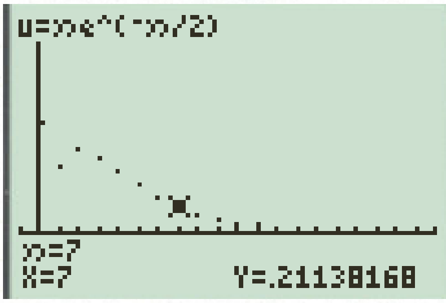
Let $f(x) = xe^{-\frac{x}{2}}$, with $f(n) = ne^{-\frac{n}{2}} = a_n$.

Consider $f'(x) = x \cdot \frac{d}{dx}[e^{-\frac{x}{2}}] + (e^{-\frac{x}{2}}) \cdot \frac{d}{dx}[x]$

$f(x) = x \cdot (e^{-\frac{x}{2}}) \cdot (-\frac{1}{2}) + (e^{-\frac{x}{2}}) \cdot (1)$

$\boxed{f(x) = e^{-\frac{x}{2}} \cdot (-\frac{x}{2} + 1)}$

The first derivative will tell us where f(x) is increasing & decreasing.



```
u=ne^(-n/2)
n=7
X=7              Y=.21138168
```

Find critical numbers:

$0 = e^{-\frac{x}{2}} \cdot (\frac{-x}{2} + 1)$

either $e^{-\frac{x}{2}} = 0$, or $-\frac{x}{2} + 1 = 0$

False, never zero.

$1 = \frac{x}{2}$

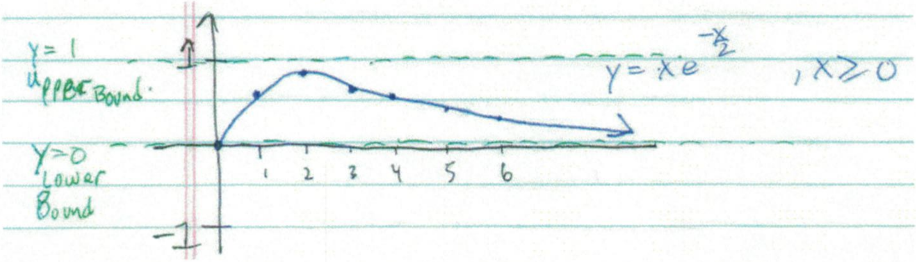
$\boxed{2 = x}$

Test the Interval $(0, \infty)$ using $x = 3$,

$f'(3) = e^{-\frac{(3)}{2}} \cdot (-\frac{(3)}{2} + 1)$

$f'(3) = \frac{1}{e^{3/2}} \cdot (-\frac{1}{2})$

$f(3) < 0$, and we can see that f(x) is decreasing for $x > 2$. This means f(x) is monotonic for $x > 2$. Is f(x) bounded for $x > 2$?



From the graph of $y = xe^{\frac{-x}{2}}$, for $x \geq 0$, we can see that the function is bounded above by $y = 1$ and bounded below by $y = 0$. Therefore, by Theorem 9.5, $\left\{ne^{\frac{-n}{2}}\right\}$ is a convergent sequence, since $\left\{ne^{\frac{-n}{2}}\right\}$ is bounded and monotonic for $n \geq 2$.

Ex. 14: *The Fibonacci Sequence*
Consider the sequence is defined by $a_{n+2} = a_{n+1} + a_n$ with $a_1 = 1$ and $a_2 = 1$.

$\{a_n\} = \{1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ 21,\ 34,\ 55,\ \ldots..\}$

This is the Fibonacci Sequence.

- Consider the following ratio of consecutive terms:

let $\dfrac{a_{n+1}}{a_n} = b_n$

$$\frac{a_{n+2}}{a_{n+1}} = \frac{a_{n+1}}{a_{n+1}} + \frac{a_n}{a_{n+1}}$$

$$\frac{a_{n+2}}{a_{n+1}} = 1 + \frac{1}{\left(\frac{a_{n+1}}{a_n}\right)}$$

$$b_{n+1} = 1 + \frac{1}{b_n}$$

– Suppose that $\lim_{n \to \infty} b_n = r$ exists ***, $r \neq 0$.

then we can also see that $\lim_{n \to \infty} b_{n+1} = \lim_{b \to \infty} b_n$

and $\lim_{n \to \infty} b_{n+1} = r$.

Therefore $\lim_{n \to \infty} b_{n+1} = \lim_{n \to \infty} \left(1 + \frac{1}{b_n}\right)$

$r = 1 + \frac{1}{r}$ ← Now, solve for r.

$r^2 = r + 1$

$r^2 - r - 1 = 0$

$r = \frac{1 \pm \sqrt{5}}{2} \rightarrow \boxed{r = \frac{1 + \sqrt{5}}{2}}$

$\frac{1 + \sqrt{5}}{2}$ is the "Golden Ratio."

*** $\lim_{n \to \infty} b_n = r$ ← you can try to justify the existance of this limit